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An extremal property of Chebyshev polynomials

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Abstract — It is proved that if $f(x)$ is a polynomial of the degree not exceeding n with real coefficients, which satisfies the condition $|f(x)| \leqslant 1$ for all $x \in [-1,1]$, then the sum of the absolute values of its coefficients attains its maximal value at $f(x) = T_n(x) = \cos(n \arccos x)$.

1. Introduction and preliminary information

Let $[\alpha,\beta]$ $(\alpha < \beta)$ be some fixed segment of the real axis, and $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a polynomial of the degree not exceeding n with real coefficients, $n \geqslant 1$. The value

$$M(f) = \max_{x\in[\alpha,\beta]} |f(x)|$$

is called the deviation of the polynomial $f(x)$ from zero on the segment $[\alpha,\beta]$.

In [5] V. A. Markov stated and studied the following general problem: for a given linear form $l(f)$ of the coefficients of a polynomial $f(x)$ of the degree not greater than n, one has to find the maximum

$$\max_{M(f)\leqslant 1} |l(f)|$$

and the corresponding extremal polynomials $f(x)$. A particular case of this problem is of special interest, this is the case when

$$l(f) = f^{(k)}(\xi)$$

where an integer number k, $0 \leqslant k \leqslant n$, and a real number ξ are fixed. Since

$$a_k = \frac{f^{(k)}(0)}{k!}$$

this particular case includes the problem of determination of the maximal absolute value of the kth coefficient of the polynomial $f(x)$ under the restriction $M(f) \leqslant 1$.

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For $k=0$ and $k=n$ the particular case of the problem of V. A. Markov indicated above was previously considered by P. L. Chebyshev [3], and for $k=1$ is was considered by A. A. Markov [4]. V. A. Markov [5] presented a solution for an arbitrary k. In particular, he had proved the following theorem.

Theorem 1.1. *If the deviation from zero on a segment $[\alpha,\beta]$ for a polynomial $f(x)$ of the degree not greater than n does not exceed 1, then the deviation of its kth derivative $f^{(k)}(x)$ from zero on the same segment does not exceed*

$$\left[\frac{\mathrm{d}^k T_n((2x-\alpha-\beta)/(\beta-\alpha))}{\mathrm{d}x^k}\right]_{x=\beta}$$

and attains this limit only for

$$f(x) = \pm T_n\left(\frac{2x-\alpha-\beta}{\beta-\alpha}\right). \tag{1.1}$$

Here and further $T_n(x)$ is the Chebyshev polynomial of the first kind,

$$T_n(x) = \cos(n\arccos x).$$

A simplified proof of this difficult theorem was given later by S. N. Bernstein [2]. Note that in the case $\xi \notin (\alpha,\beta)$ the solution to the problem of determination of the maximum

$$\max_{M(f)\leqslant 1} |f^{(k)}(\xi)|$$

is quite elementary, and the same polynomials (1.1) turn out to be extremal in this problem (see, e.g., [1]); below we present one of possible elementary proofs (see Theorem 2.1).

For the segment $[\alpha,\beta]=[-1,1]$, V. A. Markov also proved the following theorem.

Theorem 1.2. *Among all polynomials $f(x)$ of the degree not greater than n, whose deviation from zero on the segment $[-1,1]$ does not exceed 1, the following and only those polynomials (except for the case $k=0$ when any polynomial $f(x)$ with $f(0)=\pm 1$ is extremal) have the maximal absolute value of the kth coefficient:*

$$f(x) = \begin{cases} \pm T_n(x), & k \equiv n \pmod 2 \\ \pm T_{n-1}(x), & k \not\equiv n \pmod 2. \end{cases}$$

The inequalities following from this theorem for the coefficients of a polynomial $f(x)$ of the degree not greater than n with the deviation from zero on the segment $[-1,1]$ not greater than 1 are called inequalities of V. A. Markov. These inequalities can be also justified by elementary methods (see, e.g., [1]).

For a polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, we will define

$$L(f) = \sum_{k=0}^{n} |a_k|.$$

In this paper we study the problem of determination of the maximum

$$\max_{M(f)\leqslant 1} L(f)$$

and extremal polynomials $f(x)$ attaining this maximum.

If $0 \notin (\alpha,\beta)$, then one may easily show that the maximal value of $L(f)$ is attained at polynomials (1.1) only (see Corollary 2.1 from Theorem 2.1). If $0 \in (\alpha,\beta)$, then, generally speaking, polynomials (1.1) are not extremal.

Example 1.1. Let $[\alpha,\beta] = [-\gamma,\gamma]$, where $\gamma > 1$, and $n = 2$. Then

$$L(f) = \frac{\gamma^2+2}{\gamma^2} \tag{1.2}$$

for the polynomial $f(x) = T_2(x/\gamma) = 2x^2/\gamma^2 - 1$. At the same time, as is not difficult to verify,

$$\max_{M(f)\leqslant 1} L(f) = \frac{\gamma^2+2\gamma+3}{\gamma^2+2\gamma-1}$$

which is greater than (1.2). For example, the polynomial

$$f(x) = \frac{2(\gamma+1)^2}{(\gamma^2+2\gamma-1)^2}x^2 + \frac{4(\gamma^2-1)}{(\gamma^2+2\gamma-1)^2}x - \frac{\gamma^4+4\gamma^3-1}{(\gamma^2+2\gamma-1)^2}$$

is extremal here. Note that this polynomial coincides with $T_2(x)$ for $\gamma = 1$.

The main result of the paper consists in the proof of the extremal property of Chebyshev polynomials $T_n(x)$ in the problem of the maximal value of $L(f)$ for the segment $[\alpha,\beta] = [-1,1]$ (see Theorem 2.2). The proof presented here is elementary and contains a proof of the classic inequalities of V. A. Markov mentioned above.

2. Extremal property of Chebyshev polynomials

As was indicated above, the problem of determination of the maximal absolute value of the kth derivative of a polynomial $f(x)$ at the point $x = \xi$ under the restriction $M(f) \leqslant 1$ may easily be solved if $\xi \notin (\alpha,\beta)$. For convenience of the reader, below we present one of such elementary solutions.

Theorem 2.1. *For real* $\xi \notin (\alpha, \beta)$ *and integer* k, $0 \leqslant k \leqslant n$, *the maximum*

$$\max_{M(f)\leqslant 1} |f^{(k)}(\xi)|$$

is attained at polynomials (1.1) *and only at those polynomials (except for the case when* $k = 0$ *and* ξ *coincide with one of the ends of the segment* $[\alpha, \beta]$*).*

Proof. Assuming $\xi = 0$ and $0 \leqslant \alpha < \beta$, reformulate the problem in the following form: one has to determine the maximal absolute value of the kth coefficient of a polynomial $f(x)$ if $M(f) \leqslant 1$ and also to determine the corresponding extremal polynomials.

For $j = 0, 1, \ldots, n$ assume

$$x_j = \frac{(\beta-\alpha)\cos(\pi j/n) + \alpha + \beta}{2}, \quad y_j = f(x_j).$$

Here $x = \cos(\pi j/n)$, $j = 0, 1, \ldots, n$, are the points of the maximal deviation from zero for the polynomial $T_n(x)$. Since $x_j \in [\alpha, \beta]$, then $|y_j| \leqslant 1$ for all $j = 0, 1, \ldots, n$. Using the Lagrange interpolation formula, one has

$$f(x) = \sum_{j=0}^{n} \frac{(x-x_0)\cdots(x-x_{j-1})(x-x_{j+1})\cdots(x-x_n)}{(x_j-x_0)\cdots(x_j-x_{j-1})(x_j-x_{j+1})\cdots(x_j-x_n)} y_j.$$

Let X be the set $(x_0, \ldots, x_n)$ and X_j be the set X without the element x_j. Let also $\sigma_s(Y)$ be the sth elementary symmetric polynomial of the elements of a set Y (we assume $\sigma_0(Y) = 1$ and $\sigma_s(Y) = 0$ for s exceeding the number of elements in the set Y). Then

$$a_k = (-1)^{n-k} \sum_{j=0}^{n} \frac{\sigma_{n-k}(X_j)}{\omega_j} y_j$$

where $\omega_j = (x_j - x_0)\cdots(x_j - x_{j-1})(x_j - x_{j+1})\cdots(x_j - x_n)$. Thus, we have

$$|a_k| = \left| \sum_{j=0}^{n} \frac{\sigma_{n-k}(X_j)}{\omega_k} y_j \right| \leqslant \sum_{j=0}^{n} |v_{kj}|$$

where the notation

$$v_{kj} = \frac{\sigma_{n-k}(X_j)}{\omega_j}$$

is used.

Now let us show that the obtained upper bound for $|a_k|$ is attained only on polynomials (1.1). Since

$$\beta = x_0 > x_1 > \cdots > x_n = \alpha \geqslant 0$$

then $\operatorname{sign}\sigma_{n-k}(X_j) = 1$ and $\operatorname{sign}\omega_j = (-1)^j$. Therefore,

$$\operatorname{sign} v_{kj} = (-1)^j, \quad 0 \leqslant k, j \leqslant n.$$

In particular, the equality

$$|a_k| = \sum_{j=0}^{n} |v_{kj}|$$

is valid if and only if

$$y_j = \pm(-1)^j, \quad j = 0, 1, \ldots, n$$

where the same sign is chosen for all j. It remains to demonstrate that the polynomials $f(x)$ obtained for such values of y_j satisfy the condition $M(f) \leqslant 1$. Those obviously are polynomials (1.1), because the indicated condition is valid for them.

Theorem 2.1 evidently implies the following corollary.

Corollary 2.1. If $0 \notin (\alpha, \beta)$, then the maximum

$$\max_{M(f)\leqslant 1} L(f)$$

is attained at polynomials (1.1) and only at them.

The assertion of the corollary is also valid for the segment $[-1,1]$ being 'native' for the Chebyshev polynomials $T_n(x)$.

Theorem 2.2. *If $[\alpha,\beta] = [-1,1]$, then the maximum*

$$\max_{M(f)\leqslant 1} L(f)$$

is attained at polynomials $f(x) = \pm T_n(x)$ and for $n > 1$ only at them.

Proof. Use the notations introduced above in the proof of Theorem 2.1 and assume $\alpha = -1$ and $\beta = 1$.

First of all note that $\omega_j = \omega'(x_j)$, where

$$\omega(x) = \prod_{j=0}^{n} (x - x_j).$$

Since in the considered case we have $x_{n-j} = -x_j$ for all j, then

$$\omega_{n-j} = (-1)^n \omega_j, \quad \sigma_{n-k}(X_{n-j}) = (-1)^{n-k} \sigma_{n-k}(X_j).$$

Therefore,

$$a_k = (-1)^{n-k} \sum_{j=0}^{[n/2]} \frac{\sigma_{n-k}(X_j)}{\omega_j} (y_j + (-1)^k y_{n-j}), \quad k = 0, 1, \ldots, n.$$

Further assume that n is odd, $n = 2m+1$. Then for any $l = 0, 1, \ldots, m$ we have

$$\begin{aligned} a_{2l+1} &= \sum_{j=0}^{m} \frac{\sigma_{2m-2l}(X_j)}{\omega_j} (y_j - y_{2m-j+1}) \\ a_{2l} &= -\sum_{j=0}^{m} \frac{\sigma_{2m-2l+1}(X_j)}{\omega_j} (y_j + y_{2m-j+1}). \end{aligned}$$

Show that

$$|\sigma_{2m-2l+1}(X_j)| \leqslant |\sigma_{2m-2l}(X_j)|. \tag{2.1}$$

In fact,

$$0 = \sigma_{2m-2l+1}(X) = \sigma_{2m-2l+1}(X_j) + x_j \sigma_{2m-2l}(X_j)$$

which implies

$$|\sigma_{2m-2l+1}(X_j)| = |x_j||\sigma_{2m-2l}(X_j)| \leqslant |\sigma_{2m-2l}(X_j)|$$

because $|x_j| \leqslant 1$. Now, since

$$A|y' - y''| + B|y' + y''| \leqslant 2 \max\{A, B\}$$

for $|y'| \leqslant 1$ and $|y''| \leqslant 1$, we have

$$|a_{2l+1}| + |a_{2l}| \leqslant 2 \sum_{j=0}^{m} \frac{|\sigma_{2m-2l}(X_j)|}{|\omega_j|}. \tag{2.2}$$

Note that

$$\begin{aligned} \sigma_{2m-2l}(X_j) &= \sigma_{2m-2l}(X_{j,2m-j+1}) + x_{2m-j+1} \sigma_{2m-2l-1}(X_{j,2m-j+1}) \\ &= \sigma_{2m-2l}(X_{j,2m-j+1}) = (-1)^{m-l} \sigma_{m-l}(x_0^2, \ldots, x_{j-1}^2, x_{j+1}^2, \ldots, x_m^2) \end{aligned}$$

where $X_{j,2m-j+1}$ is the set X without the elements x_j and x_{2m-j+1}. Thus,

$$\operatorname{sign} \sigma_{2m-2l}(X_j) = (-1)^{m-l}.$$

Therefore, the sum

$$|a_{2l+1}| + |a_{2l}|$$

does not exceed the absolute value of the coefficient at x^{2l+1} of the Chebyshev polynomial $T_{2m+1}(x)$, which is obtained for

$$y_j = (-1)^j, \quad j = 0, 1, \ldots, 2m+1.$$

Indeed, this coefficient is equal to

$$2\sum_{j=0}^{m}\frac{\sigma_{2m-2l}(X_j)}{\omega_j}(-1)^j=(-1)^{m-l}\left[2\sum_{j=0}^{m}\frac{|\sigma_{2m-2l}(X_j)|}{|\omega_j|}\right].$$

Be summing inequalities (2.2) for all $l=0,1,\dots,m$, we get the desired assertion on the extremality of the polynomials $\pm T_{2m+1}(x)$. There are no other extremal polynomials for $m>0$, because inequality (2.1) is strict for $j>0$.

The proof for the case of an even $n=2m$ may be performed similarly.

Remark 2.1. Let $p\geqslant 1$. Since

$$|a|^p+|b|^p\leqslant(|a|+|b|)^p$$

and nonzero coefficients of the polynomial $T_n(x)$ alternate with zero ones, the assertion of Theorem 2.2 remains valid under the replacement of $L(f)$ by

$$L_p(f)=\sum_{k=0}^{n}|a_k|^p.$$

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